

Supersymmetric partner chirping of Newtonian free dampingH. Rosu^{a1}, J. L. Romero^b and J. Socorro^{a2}^a Instituto de Física de la Universidad de Guanajuato, Apdo Postal E-143,
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Summary. - We connect the classical free damping cases by means of Rosner's construction in supersymmetric quantum mechanics. Starting with the critical damping, one can obtain in the underdamping case a chirping of instantaneous physical frequency $\omega^2(t) \propto \omega_u^2 \text{sech}^2(\omega_u t)$, whereas in the overdamped case the "chirping" is of the (unphysical) type $\omega^2(t) \propto \omega_o^2 \sec^2(\omega_o t)$, where ω_u and ω_o are the underdamped and overdamped frequency parameters, respectively.

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The techniques of supersymmetric quantum mechanics gave a new impetus to many research areas in the last two decades [1]. In this note, we start with the critical free damping case in classical mechanics and construct the corresponding supersymmetric partners. The free damping Newton equation reads

$$m \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0 . \quad (1)$$

Using the gauge change of dependent variable $x = y \exp(-\frac{\gamma}{2m}t)$, this equation can be put in the following Schrödinger form in the time domain

$$y'' - \left[\left(\frac{\gamma}{2m} \right)^2 - \frac{k}{m} \right] y = 0 , \quad (2)$$

or $y'' - \omega_d^2 y = 0$, where $\omega_d^2 = (\gamma/2m)^2 - k/m$. Thus, one can discuss separately, the classical cases of underdamping (oscillating relaxation), critical damping (rapid nonoscillating relaxation), and overdamping (slow nonoscillating relaxation), i.e., $-\omega_u^2 = \omega_d^2 < 0$, $\omega_c^2 = \omega_d^2 = 0$ and $\omega_o^2 = \omega_d^2 > 0$, respectively [2]. Notice that the physical ω_d^2 frequencies are negative, the positive ones are only convenient mathematical symbols corresponding in fact to nonoscillating regimes.

We now proceed with the supersymmetric scheme that we apply in a manner similar to Rosner [3]. Thus, we start with the case of critical damping corresponding in quantum mechanics to a potential which is zero and relate it to a Schrödinger-type equation in the time domain which has a fundamental frequency at $-\omega_u^2$, i.e., a single oscillating relaxation mode that we consider as the equivalent of a bound state in the usual quantum mechanics. In other words, we solve the "fermionic" Riccati equation

$$W_1^2 - W_1' + \omega_d^2 = 0 , \quad (3)$$

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i.e.,

$$W_1^2 - W_1' - \omega_u^2 = 0 , \quad (4)$$

to find Witten's superpotential $W_1(t) = -\omega_u \tanh[\omega_u t]$ and next go to the "bosonic" Riccati equation

$$W_1^2 + W_1' + \omega_1^2(t) - \omega_u^2 = 0 , \quad (5)$$

in order to get $\omega_1^2(t) = -2\omega_u^2 \text{sech}^2[\omega_u t]$. Moreover, one can write the Schrödinger equation corresponding to the "bosonic" Riccati equation as follows

$$-\tilde{y}'' + \omega_1^2(t)\tilde{y} = -\omega_u^2\tilde{y} , \quad (6)$$

with the localized solution $\tilde{y} \propto \omega_u \text{sech}(\omega_u t)$. The physical picture is that of a chirping sech soliton profile containing a single oscillating relaxation mode self-trapped at $-\omega_u^2$ within the frequency pulse. One can employ the scheme recursively to get several oscillating relaxation modes embedded in the chirping frequency profile. Indeed, suppose we would like to introduce N oscillating relaxation modes of the type $\omega_n^2 = -n^2\omega_u^2$, $n = 1, \dots, N$ in the sech chirp. Then, one has to solve the sequence of equations

$$W_n^2 - W_n' = \omega_{n-1}^2 + n^2\omega_u^2 \quad (7a)$$

$$W_n^2 + W_n' = \omega_n^2 + n^2\omega_u^2 \quad (7b)$$

inductively for $n = 1 \dots N$ [3]. The chirp frequency containing N underdamped frequencies $n^2\omega_u^2$, $n = 1 \dots N$ is of the form $\omega_N^2(t) = -N(N+1)\omega_u^2 \text{sech}^2(\omega_u t)$. The relaxation modes can be written in a compact form as follows

$$\tilde{y}_n(t; N) \approx A^\dagger(t; N)A^\dagger(t; N-1)A^\dagger(t; N-2)\dots A^\dagger(t; N-n+2)\text{sech}^{N-n+1}\omega_u t , \quad (8)$$

i.e., by applying the first-order operators $A^\dagger(t; a_n) = -\frac{d}{dt} - a_n\omega_u \tanh(\omega_u t)$, where $a_n = N - n$, onto the "ground state" underdamped mode. This scheme can be easily generalized to embedding frequencies of the type $-\omega_{u,i}^2 = (\gamma_i/2m)^2 - k/m$ and moreover, to the construction of chirp profiles having a given continuous spectrum of relaxational modes but we shall not pursue this task here.

On the other hand, in the case of overdamping the "fermionic" Riccati equation

$$W_1^2 - W_1' + \omega_o^2 = 0 \quad (9)$$

leads to the solution $W_1 = \omega_o \tan(\omega_o t)$ and from the "bosonic" Riccati equation

$$W_1^2 + W_1' + \omega_1^2 + \omega_o^2 = 0 , \quad (10)$$

one will find $\omega_1^2(t) = 2\omega_o^2 \sec^2(\omega_o t)$. Consequently, the Schrödinger equation

$$-\tilde{y}'' + \omega_1^2(t)\tilde{y} = \omega_o^2\tilde{y} \quad (11)$$

has solutions of the type $\tilde{y} \propto \omega_o \sec(\omega_o t)$, and therefore the approach leads to unphysical results.

Referring again to the underdamped case, we also remark that an interesting, polar analysis of the chirp frequency profile can be performed by means of the change of variable $t = \ln(\tan \frac{\theta}{2})$ leading to an associated Legendre equation in the spherical polar coordinate θ

$$\frac{d^2\tilde{y}}{d\theta^2} + \cot\theta \frac{d\tilde{y}}{d\theta} + \left[N(N+1) - \frac{n^2}{\sin^2\theta} \right] \tilde{y} = 0 . \quad (12)$$

Finally, we mention that there may be potential applications of supersymmetric approaches to chirping phenomena in many areas, such as semiconductor laser physics [4], the propagation of chirped optical solitons in fibers [5], and optimal control of quantum systems by chirped pulses [6].

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